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## ESSENTIALLY HYPONORMAL OPERATORS WITH ESSENTIAL SPECTRUM CONTAINED IN A CIRCLE

SHQIPE I. LOHAJ - MUHIB R. LOHAJ

In this paper two results are given . It is proved that if the essential spectrum  $\sigma(\pi(T))$  of the bounded hyponormal operator  $T$  is contained in a circle, then  $T$  is essentially normal operator. Based on this result it is proved that if  $T \in L(H)$  with  $\text{ind } T = 0$  then  $T = \lambda U + K$  (where  $\lambda \in \mathbf{R}^+$ ,  $U$  is a unitary operator and  $K$  is a compact operator) if and only if  $TT^*$  is quasi-diagonal with respect to any sequence  $\{P_n\}$  in  $PF(H)$  such that  $P_n \rightarrow I$ , strongly.

### 1. Introduction

Let  $L(H)$  be the algebra of all bounded linear operators acting on a separable Hilbert space  $H$  and let  $K(H)$  be the ideal of all compact operators. Denote by  $A(H) = L(H)/K(H)$  the coset and let  $\pi : L(H) \rightarrow A(H)$  be the canonical projection.  $A(H)$  is a Banach algebra with respect to the norm  $\|\pi(T)\| = \inf\{\|T - K\| : K \in K(H)\}$ .  $\pi$  is a continuous linear map and  $A(H)$  is a  $C^*$ -algebra with respect to the involution  $*$  :  $\pi(T) \rightarrow [\pi(T)]^* = \pi(T^*)$ , called Calkin Algebra.

We say that an operator  $T \in L(H)$  is Fredholm if  $\pi(T)$  is invertible element in Calkin algebra  $A(H)$ . Denote by  $F_0(H)$  the set of all Fredholm operators and  $\sigma(\pi(T)) = \{\lambda \in \mathbf{C} : T - \lambda I \notin F(H)\}$  the spectrum of element  $\pi(T)$  in

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$A(H)$  which is called the essential spectrum of operator  $T$ . Further, we say that an operator  $T$  is essentially unitary operator, if  $\pi(T)$  is unitary element in  $A(H)$ . An operator is said to be hyponormal (essentially hyponormal) if  $TT^* \leq T^*T$  ( $\pi(TT^*) \leq \pi(T^*T)$ ).

## 2. Main Results

**Theorem 2.1.** *Let  $T \in L(H)$  be an essentially hyponormal operator. If the essential spectrum  $\sigma(\pi(T))$  of the operator  $T$  is contained in a circle then  $T$  is an essentially normal operator.*

Before proving Theorem 2.1 we need to prove the following result:

**Proposition 2.2.** *Let  $A$  be  $C^*$ -algebra and  $a \in A$  an invertible element, then*

$$\|a\| \cdot \|a^{-1}\| = 1 \Leftrightarrow a = \lambda u, \quad \lambda \in \mathbf{R}^+$$

where  $u$  is a unitary element in  $A$ .

*Proof.* Suppose that  $\|a\| \cdot \|a^{-1}\| = 1$ . Since  $aa^*$  is a positive self-adjoint element in algebra  $A$ , its spectrum  $\sigma(aa^*) \subset [0, \infty)$ . Further,  $\|a\|^2 = \|aa^*\|$  and the spectral radius  $r(aa^*) = \|aa^*\| = \|a\|^2$ . Now

$$\sigma(aa^*) \subseteq \left\{ \lambda \in \mathbf{R} : 0 \leq \lambda \leq \|a\|^2 \right\} \quad (1)$$

On the other hand,  $(aa^*)^{-1}$  is also positive self-adjoint element and because  $\|a\| \cdot \|a^{-1}\| = 1$  we will have

$$\sigma((aa^*)^{-1}) \subseteq \left\{ \lambda \in \mathbf{R} : 0 \leq \lambda \leq \frac{1}{\|a\|^2} \right\} \quad (2)$$

(since  $r((aa^*)^{-1}) = \frac{1}{\|a\|^2}$ ). From the above inclusions (1) and (2) we obtain  $\sigma(aa^*) = \{\|a\|^2\}$ . Since the spectrum of element  $aa^*$  contains only one point, we conclude that  $aa^* - \|a\|^2 e = q$  is a quasi-nilpotent element in  $A$ . Moreover,  $q$  is a self adjoint element in  $A$ , which implies  $q = 0$ . Therefore,  $aa^* = \|a\|^2 e$  and  $a/\|a\| = u$  is a unitary element in  $C^*$ -algebra  $A$  by which we have proved the necessary condition of the proposition.

The proof of the sufficient condition of the proposition is trivial.  $\square$

**Remark 2.3.** Since  $L(H)$  is a  $C^*$ -algebra from the Proposition 2.2 we observe that every invertible operator  $T \in L(H)$  that satisfies the property  $\|T\| \cdot \|T^{-1}\| = 1$  is a scalar multiple of some unitary operator  $U \in L(H)$ .

Proof of Theorem 2.1. Let  $T$  be an essentially hyponormal operator. It is easy to see that the translation and inverse of essentially hyponormal operator is essentially hyponormal. For this reason without loss of generality suppose that

$$\sigma(\pi(T)) \subseteq \{\lambda \in \mathbf{C} : |\lambda| = l\}$$

Then,  $r(\pi(T)) = l$  and  $r(\pi(T)^{-1}) = \frac{1}{l}$ . Further, since the element  $\pi(T)$  is normal in the Calkin algebra, we will have  $r(\pi(T)) = \|\pi(T)\|$  (see [3] ) and therefore

$$\|\pi(T)\| \cdot \|\pi(T)^{-1}\| = 1$$

Now, by Proposition 1 and the latter relation we conclude that the operator  $T$  is essentially normal. By which we have completed the proof of Theorem 2.1.

Further on, let  $(QD)\{P_n\}$  be a class of quasidiagonal operators such that the quasidiagonality is assumed with a common sequence of orthogonal projections  $\{P_n\}$  (see[6] , [7] ). Then we can give the following result:

**Proposition 2.4.** *Let  $T \in L(H)$  with  $\text{ind}T = \dim \text{Ker}T - \dim \text{Ker}T^* = 0$  then*

$$TT^* \in \bigcap_{\substack{\{P_n\} \subset PF(H) \\ P_n \xrightarrow{s} I}} (QD)\{P_n\} \Leftrightarrow T = \lambda U + K$$

where  $\lambda \in \mathbf{R}^+$  ,  $U$  is a unitary operator and  $K$  is a compact operator.

*Proof.* Let

$$TT^* \in \bigcap_{\substack{\{P_n\} \subset PF(H) \\ P_n \xrightarrow{s} I}} (QD)\{P_n\}.$$

It means that  $TT^* \in (QD)\{P_n\}$  for every sequence  $\{P_n\}$  in  $PF(H)$  such that  $P_n \rightarrow I$ , strongly. Hence, operator  $TT^*$  is uniformly quasidiagonal and thus  $TT^*$  is thin(see [5]). Therefore,  $TT^* = \alpha I + K$ , where  $\alpha \geq 0$  and  $K \in K(H)$  . Further,

$$\pi(TT^*) = \pi(T)\pi(T^*) = \alpha\pi(I)$$

Hence  $\pi(T)/\sqrt{\alpha}$  is a unitary element in Calkin Algebra  $L(H)/K(H)$  . Since  $\text{ind} T = 0$ , there exists a unitary operator  $U \in L(H)$ , and a compact operator  $K \in L(H)$  such that  $T/\sqrt{\alpha} = U + K$  (see Theorem 3.1 in [2]).

Conversely. If  $T = \lambda U + K$  then

$$TT^* = (\lambda U + K)(\bar{\lambda}U^* + K^*) = |\lambda|^2 I + \lambda U K^* + \bar{\lambda} K U^* + K K^* = |\lambda|^2 I + K_1,$$

where  $K_1 \in K(H)$ , which completes the proof of Proposition 2.4, because the operators  $I, K_1 \in (QD)\{P_n\}$ , for every sequence  $\{P_n\}$  in  $PF(H)$ , such that  $P_n \rightarrow I$ , strongly.  $\square$

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SHQIPE I. LOHAJ

Electronic Faculty

University of Prishtina

e-mail: shqipe\_lohaj@yahoo.com

MUHIB R. LOHAJ

Faculty of Natural Science

University of Prishtina

e-mail: muhib\_lohaj@yahoo.com